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THE SOLUTION OF THE SECOND FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A PLATE WITH A DOUBLY SYMMETRIC TWO-CUSP CUT†

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As an extension of a previous paper [1], devoted to solving the first fundamental and mixed problems for a plate with a doubly symmetric two-cusp cut, the second fundamental problem for a plate with the same cut is solved by the same method.

1. THE SOLUTION PROCEDURE

The second boundary-value problem for an unbounded region D consists of finding two functions that are analytic in $D\backslash\{\infty\}$ [2]

$$f(z)=\Gamma z-\frac{X+iY}{2\pi(1+\varkappa)}\ln z+\frac{a}{z}+\ldots,\quad g(z)=\Gamma' z+\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln z+\frac{a'}{z}+\ldots \tag{1.1}$$

where Γ, Γ' and $X+iY$ are the known constants. The boundary condition has the form

$$\{\varkappa f(z)-z\overline{f'(z)}-\overline{g(z)}\}|_{z=z(t)}=2\mu(u(t)+iv(t)) \tag{1.2}$$

where $z=z(t)$, $t\in[0,l]$ is the equation of the boundary curve ∂D. We will assume that $u'(t)$ and $v'(t)$ are Hölder's functions.

We will change to the function $z(\zeta)$, which conformally maps the region $E^-=\{\zeta=\xi+i\eta, |\zeta|>1\}$ into D with corresponding $z(\infty)=\infty$ and differentiate both sides of equality (1.2) with respect to t. We obtain the boundary condition

$$\{\varkappa\overline{\Phi(\zeta)}\overline{z'(\zeta)}-\overline{z'(\zeta)}\,\Phi(\zeta)+\overline{z(\zeta)}\Phi'(\zeta)\zeta^2+\Psi(\zeta)z'(\zeta)\zeta^2\}|_{\zeta=e^{i\theta}}=Q(\theta) \tag{1.3}$$

$$Q(\theta)=2i\mu[u'(t)-iv'(t)]|_{t=t(\theta)}\,e^{i\theta}\,|z'(e^{i\theta})|,\quad \theta\in[-\pi,\pi] \tag{1.4}$$

where the functions $\Phi(\zeta)=f'(z(\zeta))$ and $\Psi(\zeta)=g'(z(\zeta))$ are analytic in E^-.

If $q(\zeta)=\bar{z}(\zeta^{-1})$ and $r(\zeta)=\bar{z}'(\zeta^{-1})$ are meromorphic in E^-, we can restore the meromorphic functions

$$K_j(\zeta)=(-1)^j\varkappa\Phi(\zeta)z'(\zeta)-r(\zeta)\Phi(\zeta)+q(\zeta)\Phi'(\zeta)\zeta^2+\Psi(\zeta)z'(\zeta)\zeta^2,\quad j=1,2 \tag{1.5}$$

using the boundary values $\operatorname{Re}K_1(\zeta)$ ($\operatorname{Re}Q(\theta)$) and $\operatorname{Im}K_2(\zeta)$($\operatorname{Im}Q(\theta)$), (1.3) and (1.4). Further, $\Phi(\zeta)$ and $\Psi(\zeta)$ are restored by $K_j(\zeta)$ (1.5).

2. SOLUTION OF THE PROBLEM

The function

$$z(\zeta)=\frac{i(b^2\zeta^2+1)}{\zeta(b^2-1)}+\frac{\zeta(b^2-1)}{i(b^2\zeta^2+1)},\quad b>1 \tag{2.1}$$

maps E^- into D. Using Eq. (1.5), we obtain

$$2\varkappa\Phi(\zeta)z'(\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Q(\theta)}\frac{\zeta+e^{i\theta}}{\zeta-e^{i\theta}}d\theta+A-\frac{2}{\zeta^2}\left[\frac{i}{b^2-1}\overline{\Gamma}-\frac{ib^2}{b^2-1}\overline{\Gamma'}\right]-$$

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$$-\frac{\varkappa(X+iY)}{\pi(1+\varkappa)\zeta}-2\overline{B}\frac{\zeta-ib}{1+ib\zeta}-2\overline{C}\frac{\zeta+ib}{1-ib\zeta}-2\overline{D}\left(\frac{\zeta-ib}{1+ib\zeta}\right)^2-2\overline{E}\left(\frac{\zeta+ib}{1-ib\zeta}\right)^2 \tag{2.2}$$

Requiring that $\Phi(\infty) = \Gamma$, we have

$$A+\frac{2i}{b}\overline{B}-\frac{2i}{b}\overline{C}+\frac{2}{b^2}\overline{D}+\frac{2}{b^2}\overline{E}=\frac{2\varkappa ib^2}{b^2-1}\Gamma-\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Q(\theta)}\,d\theta \tag{2.3}$$

By (1.5)

$$\Psi(\zeta)=[\zeta^2z'(\zeta)]^{-1}[K_1(\zeta)-\varkappa\Phi(\zeta)z'(\zeta)+r(\zeta)\Phi(\zeta)-q(\zeta)\Phi'(\zeta)\zeta^2] \tag{2.4}$$

where the coefficients of $(\zeta \pm ib)^{-2}$ and $(\zeta + ib)^{-1}$ must vanish, which yields four equations

$$\begin{gathered}D(b^2-1)-i\Phi(ib)b^2/2=0,\quad E(b^2-1)-i\Phi(-ib)b^2/2=0\\ B+2ibD+b\Phi(ib)=0,\quad C-2ibE-b\Phi(-ib)=0\end{gathered} \tag{2.5}$$

Using formula (2.2), we obtain from (2.3) and (2.5) a system of five equations in five unknown constants

$$\begin{aligned}&A+\frac{2i}{b}B-\frac{2i}{b}C+\frac{2}{b^2}\overline{D}+\frac{2}{b^2}\overline{E}=\frac{2\varkappa ib^2}{b^2-1}\Gamma-\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Q(\theta)}d\theta\\ &D-b\beta[A-2i\delta\overline{C}+2\delta^2\overline{E}]=b\beta R_+(b)\\ &E-b\beta[A+2i\delta\overline{B}+2\delta^2\overline{D}]=b\beta R_-(b)\\ &B+2ibD+2i\beta(1-b^2)[A-2i\delta\overline{C}+2\delta^2\overline{E}]=-2i\beta(1-b^2)R_+(b)\\ &C-2ibE-2i\beta(1-b^2)[A+2i\delta\overline{B}+2\delta^2\overline{D}]=2i\beta(1-b^2)R_-(b)\end{aligned} \tag{2.6}$$

Here

$$\beta=\frac{b^3(1+b^2)^2}{4\varkappa(b^4+1)(b^4+b^2+1)},\quad \delta=\frac{2b}{b^2+1}$$

$$R_\pm(b)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Q(\theta)}\frac{\pm ib+e^{i\theta}}{\pm ib-e^{i\theta}}d\theta-\frac{2i}{b^2(b^2-1)}(\overline{\Gamma}-b^2\overline{\Gamma'})\mp\frac{\varkappa(X+iY)}{\pi ib(1+\varkappa)}$$

After determining the constants, the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ can be obtained from (2.2) and (2.4) respectively. Now we reconstruct $f(z(\zeta))$ and $g(z(\zeta))$

$$f(z(\zeta))=\int_1^\zeta\Phi(\zeta)z'(\zeta)d\zeta+f(z(1)),\quad g(z(\zeta))=\int_1^\zeta\Psi(\zeta)z'(\zeta)d\zeta+g(z(1))$$

The function $z(\zeta)$ is the same as in (2.1), $f(z(1))$ and $g(z(1))$ are the quantities that satisfy the equation

$$\varkappa f(z(1))-\overline{g(z(1))}=2\mu[u(t_0)+iv(t_0)]+z(1)\overline{\Phi(1)}$$

and t_0 is the value of the initial parameter corresponding to the point $z = z(1)$.

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